

ANY FIP REAL COMPUTES A 1-GENERIC

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ABSTRACT. We construct a computable sequence of computable reals $\langle X_i \rangle$ such that any real that can compute a subsequence that is maximal with respect to the finite intersection property can also compute a Cohen 1-generic. This is extended to establish the same result with 2IP in place of FIP. This is the first example of a classical theorem of mathematics has been found to be equivalent, both proof theoretically and in terms of its effective content, to computing a 1-generic.

1. INTRODUCTION

Of all the axioms of mathematics, perhaps the most well-known is the Axiom of Choice; which has many classically equivalent forms. These include, for example Zorn's Lemma, the Well-Ordering Principle, etc. Rubin and Rubin [7] give many such classically equivalent versions.

These equivalent forms do not, however, remain equivalent when we begin to examine them through the fine grained lenses of either computability theory or proof theory, especially via reverse mathematics. One of the classic versions of the axiom of choice is called the *finite intersection property*. In the following definition we will think of reals as being subsets of \mathbb{N} . We will often abuse notation and refer to reals interchangeably as sets or vice-versa.

Definition 1.1. *A set (or sequence) of reals has the finite intersection property if given any finite subset of the reals (or any finite set of reals that appear in the sequence) that set has a nonempty intersection.*

In [3, 4], Dzhafarov and Mummert began a systematic study of the proof-theoretical and computability-theoretical strength of various incarnations of the axiom of choice and related principles of finite character. Of concern to the present paper, Dzhafarov and Mummert initiated

Downey was supported by the Marsden Fund of New Zealand. This work was partially supported by a grant from the Simons Foundation (#315283 to Peter Cholak). Igusa was partially supported by EMSW21-RTG-0838506. Research was (partially) completed while the authors were visiting the Institute for Mathematical Sciences, National University of Singapore in 2014.

an investigation of such properties for the finite intersection principle. In this paper we will be concerned with a logical analysis of this principle.

As with Dzhafarov and Mummert's analysis, it is quite important how we present the family. An *instance* of an intersection problem is a sequence of reals, subsets of \mathbb{N} , $\mathcal{X} = \{A_i \mid i < \omega\}$. We allow repetitions in the family. There are possibly many equivalent enumerations of the same families and these may not be computationally equivalent. We will write that $\hat{\mathcal{X}} \subseteq \mathcal{X}$ if every element of the sequence $\hat{\mathcal{X}}$ is in \mathcal{X} .

Thus we will consider the following.

Definition 1.2. *Let $\mathbb{X} = \langle X_i \mid i < \omega \rangle$ be a uniformly computable sequence of computable reals, not all empty. Let $f : \mathbb{N} \rightarrow \mathbb{N}$. Then, f is FIP with respect to \mathbb{X} if $\{X_{f(i)} \mid i < \omega\}$ is maximal with respect to the finite intersection property, that is to say, $\{X_{f(i)}\}$ has the finite intersection property, and if $\{X_{f(i)}\} \subset \{X_{g(i)}\}$, then $\{X_{g(i)}\}$ does not have the finite intersection property.*

In this case, we will frequently abuse notation and say that $\langle X_{f(i)} \rangle$ or $\{X_{f(i)}\}$ is FIP.

A related principle is called $n\text{IP}^1$, and it is the same as the above, except it only seeks each n -tuple of sets to have nonempty intersection. The principle 2IP thus says that every family of sets has a maximal subfamily where each pair of sets in the family has nonempty intersection.

In [4], Dzhafarov and Mummert showed that there was a computable instance of FIP with no computable solution.

Definition 1.3 (Dzhafarov and Mummert [4]). *A degree \mathbf{a} is FIP (or FIP-bounding) if given any computable \mathbb{X} as above, \mathbf{a} can compute a function f that is FIP with respect to \mathbb{X} . Similarly we can define a degree to be $n\text{IP}$.*

It is easy to see that $\mathbf{0}'$ is FIP and $n\text{IP}$, simply by taking a computable family and making a maximal FIP subfamily one set at a time, noting that nonempty intersection of computable sets is Σ_1^0 . Dzhafarov and Mummert established each FIP degree is also $n\text{IP}$ for all n , and any $(n+1)\text{IP}$ degree is also $n\text{IP}$. One of the fundamental questions left open by Dzhafarov and Mummert was whether any of these implications are proper. One of the corollaries of the work in the present paper is that the answer is, somewhat surprisingly, no.

¹This principle was referred to as $\overline{D}_n - \text{IP}$ by Dzhafarov and Mummert. We follow the terminology of Downey, Diamondstone, Greenberg and Turetsky [2].

Dzhafarov and Mummert demonstrated that FIP has relationships with a number of well-studied principles of Reverse Mathematics and computability theory. These include cohesive sets, and the Atomic Model Theorem from model theory. Of relevance to us will be the following which relates FIP to hyperimmune and 1-generic degrees. The reader should recall that a degree \mathbf{a} is called *hyperimmune* if \mathbf{a} computes a function f which is not dominated by any computable function: if g is computable $\exists^\infty s(f(s) > g(s))$. Some authors refer to degrees which are not hyperimmune as *computably dominated* for this reason.

The reader should recall that \mathbf{a} is called 1-generic if it contains a 1-generic set G , where G is 1-generic if it is Cohen generic for 1-quantifier arithmetic. Equivalently, for all c.e. sets of strings V , there is a string $\tau \prec G$ such that either $\tau \in V$ or for all $\sigma \in V$, $\tau \not\leq \sigma$. Such sets encode the basic behaviors coming to all finite extension arguments. All 1-generic sets are hyperimmune.

Theorem 1.4 (Dzhafarov and Mummert [4]). *Every FIP degree is hyperimmune. The following are FIP degrees.*

- (1) *All nonzero c.e. degrees.*
- (2) *Degrees that are $\mathbf{0}'$ -hyperimmune.*
- (3) *Degrees that compute 1-generics which meet a prescribed sequence of dense Π_1^0 sets of strings.*

Dzhafarov and Mummert also partially classified how the FIP degrees relate to other bounding degrees such as the atomic model bounding degrees and prime model bounding degrees (see, e.g. [1, 5]).

In view of this, the precise classification of the FIP and n IP degrees seems a very intriguing problem. Recently, Diamondstone, Downey, Greenberg and Turetsky [2] made some significant advances for the classification, as well as establishing some basic results. They showed that the FIP and 2IP degrees have acceptable universal families, meaning that if a degree computes a solution to the universal family, then it will compute a solution to any family. They also proved that if G is 1-generic then the Turing degree of G is 2IP and hence FIP.² Those authors also looked at finite variations of FIP and n IP showing that the principles differed in strength for some such variations, and also looked at the situation where *infinite* intersection was asked for. In this last formulation, the precise strength needed was found. (Namely

²Note that *more* families have 2IP than FIP. For example, the family consisting of the 3 sets $\{1, 2\}$, $\{2, 3\}$ and $\{3, 1\}$ has 2IP but not FIP.

\mathbf{a} can compute a solution with infinite intersection property for all computable families iff $\mathbf{a} \geq \mathbf{0}''$).

Diamondstone, Downey, Greenberg and Turetsky [2] gave a complete answer to the question of FIP and 2IP bounding in a limited setting.

Theorem 1.5 (Diamondstone, Downey, Greenberg and Turetsky [2]). *A Δ_2^0 degree is FIP iff it is 2IP iff it computes a 1-generic.*

The goal of the present paper is to give a complete classification of the FIP and n IP degrees.

Theorem 1.6. *A degree \mathbf{a} is FIP iff it is 2IP iff it computes a 1-generic.*

In view of results in [2], it is enough to prove that if \mathbf{a} is an FIP (2IP) degree then it computes a 1-generic. We remark that Theorem 1.6 gives the first example of a classical theorem whose strength aligns exactly with the ability to construct a 1-generic.

In the proof, we will first establish the FIP case. Later we will modify this proof to establish the 2IP case. We remark that the proofs from [2] heavily used approximations and were uniform in the sense that a single procedure Φ was constructed which took the given Δ_2 solution S to the 1-generic G . Preliminary analyses show that no such uniform solution is possible for the global case. Our proof is nonuniform, and constructs infinitely many possible functionals and we will argue that at least one works.

In the final section, we will discuss the proof theory of the situation. We will do this in the context of reverse mathematics in the sense of Simpson [9]. A general reference for computability is Soare [10].

2. THE FIP CASE

In this section, we construct an $\mathbb{X} = \langle X_i \rangle$ such that any FIP f can compute a 1-generic. We will use one primary Turing functional Φ to compute a 1-generic from f , and a countable collection of secondary functionals Ψ_i to compute the halting set, K , from f . The Turing degree of K is \emptyset' , which can be used to compute a 1-generic. The outputs of the functionals will depend only on the range of f . (Intuitively, this means that the 1-generic will depend only on the FIP set, not on the FIP sequence.) We will guarantee that for any FIP f , at least one of these functionals succeeds at its task.

We produce a collection of labels that we will place onto the tree $2^{<\omega}$. Each σ in $2^{<\omega}$ will have either one label, or an infinite countable collection of labels on it. Each label will be a name for a computable real, and the sequence $\langle X_i \rangle$ will be the sequence of labels that are

used in our construction, in the order that they first appear. The construction will ensure that for any finite or infinite path through $2^{<\omega}$, the sequence of reals whose labels appear on the path has the finite intersection property (but is, perhaps, not maximal with respect to having the finite intersection property).

The labels that we will use are as follows.

We will have labels of the form A_σ , thought of as structural labels. For each $\sigma \in 2^{<\omega}$, the label A_σ will be placed on σ and in no other location. The purpose of these labels will be to help tie f to a path through $2^{<\omega}$. These labels will be what Φ uses to attempt to compute a 1-generic.

We will also have labels of the form $B_{i,j}$, thought of as coding labels. These labels will be placed on $2^{<\omega}$ in a Σ_1^0 way, although for simplicity of the construction, we will assume we are able to place a countable computable set of B labels on one node during a single stage of the construction. They will be the labels that the Ψ_i use to attempt to compute K .

These will be the only labels that we place.

At each stage, s , of the construction, for each $\sigma \in 2^{<\omega}$ of length $< s$, we choose some new $n \in \mathbb{N}$, and put n into every set whose label is on an initial segment of σ at stage s . That number n will not go into any other sets. Thus, labeling $2^{<\omega}$ will be equivalent to defining (and producing computations of) all of the sets X_i .

Throughout the construction, we will frequently wish to alternate between objects that we construct and the sets of reals whose labels appear on those objects. To this end, we make a collection of notational definitions.

First, the labels will also be constant symbols for the reals that they label. So, for instance, we begin the construction by placing the label A_λ onto the empty node λ . Since this is the first label that we use, that means the label of X_0 will be A_λ . This convention will allow us to treat A_λ as a symbol for a real (and in particular, have that $X_0 = A_\lambda$).

If $f : \mathbb{N} \rightarrow \mathbb{N}$ is a function, then $\tilde{f} = \{X_i : i \in \text{ran}(f)\}$.

A *node* in $2^{<\omega}$ is an element of $2^{<\omega}$.

A *finite path* in $2^{<\omega}$ is a sequence of successive nodes in $2^{<\omega}$. If the finite path begins at the root of $2^{<\omega}$, then it is the set of $\tau \preceq \sigma$ for some $\sigma \in 2^{<\omega}$. This finite path will also sometimes be denoted σ .

If $P \in 2^\omega$ is a real, regarded as an infinite path through $2^{<\omega}$, then \tilde{P} is the set of reals whose labels appear on initial segments of P .

If $\sigma \in 2^{<\omega}$, then $\tilde{\sigma}$ is the set of reals whose labels appear on proper initial segments of σ . This set will also be referred to as the sequence of reals along the finite path σ .

All of the functionals that we define in the construction will depend only on the range of their oracle, f . Thus, the following definition will be well defined. If θ is a Turing functional that we define during this construction (so θ is either Φ or one of the Ψ_i), and if S is an infinite subset of the reals that we construct during the construction, then $\theta^S = \theta^f$ where f is any function such that $\tilde{f} = S$.

If $\sigma \in 2^{<\omega}$, and if θ is a functional that we define during the construction, then θ^σ is the maximal finite string α such that for any $S \supseteq \tilde{\sigma}$, $\theta^S \succ \alpha$. In other words, θ^σ is the initial segment of the output of θ that is determined by σ . In the construction, our functionals will be defined such that for any $\sigma \in T$, θ^σ will always be defined.

Now that we have established this notation, note the following lemmas that can be proved entirely from the above heuristics, without appeal to the specific manner in which we decide how place our B labels onto $2^{<\omega}$.

Lemma 2.1. *Given any finite set, S , of reals from among the reals that we build, those reals have the finite intersection property if and only if there is a node $\sigma \in 2^{<\omega}$ such that $\tilde{\sigma} \supseteq S$.*

Proof. S is finite, and so, if there is such a $\sigma \in 2^{<\omega}$, then at some finite stage of the construction, the label of every real in S has appears on an initial segment of σ . At that stage (and every stage thereafter), a common element is placed into all the reals in $\tilde{\sigma} \supseteq S$.

Conversely, if S has the finite intersection property, then in particular, the reals in S must have a common intersection (because S is a finite set, and so a finite subset of itself). Thus, there must be some n that is in all of the reals. That n was chosen at some finite stage to go exactly into the reals along some $\sigma \in 2^{<\omega}$. Thus, all the reals in our set must be in $\tilde{\sigma}$. \square

Lemma 2.2. *Given any infinite path, P through $2^{<\omega}$, \tilde{P} has the finite intersection property.*

Note that this lemma does not claim that \tilde{P} is FIP with respect to the sequence of all reals that we build.

Proof. Given any finite number of reals in \tilde{P} , there is a finite path that they are all on. By Lemma 2.1, those reals have the finite intersection property, and so they intersect. \square

Lemma 2.3. *Let S be an FIP set of reals for the sequence \mathbb{X} that we build. Then there exists a unique infinite path Y through $2^{<\omega}$ such that $\tilde{Y} \subseteq S$.*

Proof. First, we prove existence.

Note first that if σ and τ are incomparable in $2^{<\omega}$, then A_σ and A_τ do not intersect, because those labels only appear on σ and τ respectively. Thus, for all A sets that are in S , the subscripts of the labels must be comparable.

Secondly, if $A_\sigma \in S$, and if $B_{i,j} \in \tilde{\sigma}$, then $B_{i,j} \in S$. This is because any finite path containing A_σ also contains $B_{i,j}$, and so by Lemma 2.1, if S has the finite intersection property, and $A_\sigma \in S$, then $S \cup \{B_{i,j}\}$ has the finite intersection property. By maximality of S , we may conclude that $S = S \cup \{B_{i,j}\}$.

It remains to show that there exist infinitely many σ such that $A_\sigma \in S$, because then Y can be the union of those σ .

Assume this is not the case, and let σ_0 be maximal such that $A_{\sigma_0} \in S$. Let $\sigma_1 = \sigma_0 \hat{\ } 0$, the left branch off of σ_0 , and let $\sigma_2 = \sigma_0 \hat{\ } 1$, the right branch off of σ_0 . Because S is maximal, and by choice of σ_0 , we have that $S \cup \{A_{\sigma_1}\}$ does not have the finite intersection property. So fix a finite set $F_1 \subseteq S$ such that $F_1 \cup \{A_{\sigma_1}\}$ has an empty intersection. Likewise, fix a finite set $F_2 \subseteq S$ such that $F_2 \cup \{A_{\sigma_2}\}$ has an empty intersection.

Let $F = F_1 \cup F_2 \cup \{A_{\sigma_0}\}$. Then F is a finite subset of S , and so it has the finite intersection property, so by Lemma 2.1, there is some σ such that $F \subseteq \tilde{\sigma}$. By construction, $A_{\sigma_0} \in F$, and so $\sigma_0 \preceq \sigma$. For $i = 1, 2$, $F \cup \{A_{\sigma_i}\}$ does not have the finite intersection property, so by Lemma 2.1, σ is incomparable to both σ_1 and σ_2 . This is a contradiction, because every extension of σ_0 is comparable to either σ_1 or σ_2 . \square

We define Φ^S to be the Y satisfying Lemma 2.3. Note that this is uniformly computable from any f such that $\tilde{f} = S$ because, to compute the first n bits of Y , we need only wait to find some σ of length n such that A_σ is in \tilde{f} .

3. CONSTRUCTION

We now describe the way that the B labels get placed onto $2^{<\omega}$, and the way that our Ψ functionals use them to compute K in the event that Φ^f is not 1-generic.

To make our notation simpler, we will assume that the c.e. sets W_i always enumerate strings in $2^{<\omega}$. Under this convention, a real Y is 1-generic if and only if, for every i , either W_i enumerates an initial

segment of Y , or there is an initial segment of Y such that W_i never enumerates any extensions of that initial segment.

The construction is as follows. At stage s of the construction, let K_s be the s th approximation to the halting set K . Then, for each $i < s$, if W_i enumerates σ in $\leq s$ steps, and if W_i has not enumerated an initial segment of σ at a previous stage, then for every j such that $j \notin K_s$, we place the label $B_{i,j}$ onto the node σ . This completes the construction.

We now define Ψ_i so that $\Psi_i^S(j) = 1$ if $j \in K$, and $\Psi_i^S(j) = 0$ if $B_{i,j} \in S$.

More formally, to determine $\Psi_i^f(j)$, search for some value of s such that one of the following holds.

- (1) $j \in K_s$
- (2) There is some $t < s$ such that the labels for X_0, \dots, X_t have been determined by stage s of the construction and $(\exists r < s)(\exists q \leq t)(f(r) = q \ \& \ \text{the label of } X_q \text{ is } B_{i,j})$.

If an s satisfying (1) is found first, then halt and output 1. If an s satisfying (2) is found first, then halt and output 0. (The primary reason for the formalism here is that the informal definition is not a definition, because both of the two “if” clauses could be true. In the conditions where Ψ_i is supposed to work, exactly one of the two “if” clauses will be true, and the informal definition will be all we need.)

This completes the labeling of $2^{<\omega}$, and thus of \mathbb{X} , and also Ψ_i for each i . We now move on to prove that the construction functions as desired.

Lemma 3.1. *Let f be FIP for \mathbb{X} . Let $Y = \Phi^f$ and assume Y is not 1-generic. Then there exists some i such that $\Psi_i^f = K$.*

To explain how this will be proved, note the following lemma which shows the only mechanism by which reals that are not in \tilde{Y} might still be able to be in \tilde{f} .

Lemma 3.2. *Let Y be a path through $2^{<\omega}$. Let $B_{i,j}$ be a B label that is not in \tilde{Y} . Then $\tilde{Y} \cup \{B_{i,j}\}$ has the finite intersection property if and only for every initial segment σ of Y , there is an extension $\tau \succeq \sigma$ such that $B_{i,j} \in \tilde{\tau}$.*

Proof. Let Y be an infinite path through $2^{<\omega}$. Let $B_{i,j}$ be a B label that is not in \tilde{Y} .

Assume that $\tilde{Y} \cup \{B_{i,j}\}$ has the finite intersection property, and let $\sigma \prec Y$. Then $A_\sigma \in \tilde{Y}$ and so it must intersect $B_{i,j}$. By Lemma 2.1,

there must be some τ such that $\{A_\sigma, B_{i,j}\} \subseteq \tilde{\tau}$. Because $A_\sigma \in \tilde{\tau}$, we have that $\tau \succeq \sigma$.

Conversely, assume that for every $\sigma \prec Y$, there exists a $\tau \succ \sigma$ such that $B_{i,j} \in \tilde{\tau}$. We must prove that $\tilde{Y} \cup \{B_{i,j}\}$ has the finite intersection property. So let F be a finite subset of $\tilde{Y} \cup \{B_{i,j}\}$. Let $F_0 = F \cap \tilde{Y}$. Because F_0 is a finite subset of \tilde{Y} , we have that there is a $\sigma \prec Y$ such that $F_0 \subseteq \tilde{\sigma}$. By assumption, extend σ to a τ such that $B_{i,j} \in \tilde{\tau}$. Then $F \subseteq \tilde{\tau}$, and so, by Lemma 2.1, F has nonempty intersection. \square

We now move on to prove Lemma 3.1

Proof. Let f be FIP for \mathbb{X} . Let $Y = \Phi^f$, the unique path through $2^{<\omega}$ such that $\tilde{Y} \subseteq \tilde{f}$. (Note that such a Y is guaranteed to exist by Lemma 2.3, and is equal to Φ^f by the comment after the proof of the lemma.)

If Y is not 1-generic, then $\tilde{Y} \neq \tilde{f}$, as follows. Let i be such that Y neither meets nor avoids W_i . Fix $j \notin K$. Then $B_{i,j}$ is not in \tilde{Y} . This is because Y does not meet W_i , and so we never put any B_i labels onto initial segments of Y .

Furthermore, every initial segment σ of Y has an extension $\tau \succeq \sigma$ such that $B_{i,j} \in \tilde{\tau}$. This is because Y does not avoid W_i , and so every initial segment of Y has an extension τ that is in W_i , and every time such a τ enters W_i , we put the label $B_{i,j}$ onto that τ . (Note, this uses the fact that $j \notin K$, and K is c.e., so we have that $\forall s \ j \notin K_s$.)

By Lemma 3.2, $\tilde{Y} \cup \{B_{i,j}\}$ has the finite intersection property, and so \tilde{Y} is not FIP, because it is not maximal. Thus, $\tilde{Y} \neq \tilde{f}$.

So now, let $B_{i_0,j_0} \in \tilde{f} \setminus \tilde{Y}$. Then we claim that $\{j : B_{i_0,j} \in \tilde{f}\} = \text{ar } K$. (No A -labeled sets can be in $\tilde{f} \setminus \tilde{Y}$, because A -labels each occur only once, so if they are not in \tilde{Y} , then there is an A -labeled node in \tilde{Y} that they do not intersect.)

To see this, first of all, note that $j_0 \notin K$. Otherwise, B_{i_0,j_0} only occurs at finitely many locations, and so by Lemma 3.2, $\tilde{Y} \cup \{B_{i_0,j_0}\}$ cannot have the finite intersection property.

Note also that there is no j such that $B_{i_0,j} \in \tilde{Y}$. This is because $j_0 \notin K$, and so, at any stage, if we put any $B_{i_0,j}$ onto any initial segment of Y , we would have placed B_{i_0,j_0} onto that same location, contradicting our choice of B_{i_0,j_0} .

If $j \notin K$, then $B_{i_0,j}$ occurs at exactly the locations where B_{i_0,j_0} occurs, and so $B_{i_0,j}$ has finite intersection with exactly the sets that B_{i_0,j_0} does, so if $B_{i_0,j_0} \in \tilde{f}$, then $B_{i_0,j} \in \tilde{f}$. (In this construction, $B_{i_0,j_0} = B_{i_0,j}$. If this is concerning to the reader, the construction could be modified so that we use the construction we present to add

odd numbers to our sets, and we use the even numbers to give each set a single element that is in no other set. This does not change our FIP sequences, and ensures that our sets are all distinguished.)

If $j \in K$, then $B_{i_0,j}$ occurs at finitely many locations, and so by Lemma 3.2, $\tilde{Y} \cup \{B_{i_0,j}\}$ does not have the finite intersection property, and so $B_{i_0,j} \notin \tilde{f}$.

Thus, $\{j : B_{i_0,j} \in \tilde{f}\} = \text{ar}K$, and so $\Psi_{i_0}^f$ is a computation of K . \square

We now prove the main result of this section.

Theorem 3.3. *Let X be a real that can compute an FIP function for any uniformly computable sequence of reals. Then X can compute a Cohen 1-generic.*

Proof. The sequence of reals that we constructed is uniformly computable. Let f be an FIP function for the sequence that we constructed. By Lemma 3.1, either Φ^f computes a 1-generic, or there is some i such that Ψ_i^f computes K . We know that K can be used to compute a 1-generic, and so it follows that in either case, f can compute a 1-generic.

Thus, if X can compute such an f , then X can compute a 1-generic. \square

4. THE 2IP CASE

In this section we will modify the proof of the previous section to work with the more delicate property of 2IP in place of FIP. The problems are caused by the B sets from the previous construction. Because they can appear multiple times on $2^{<\omega}$ in incomparable locations, it is possible for it there to be a finite set involving B sets (and potentially other sets as well) such that any two of them appear together along some path, but such that they do not all appear anywhere on $2^{<\omega}$ on a single finite path.

In this modification, we will produce a construction in which every 2IP set is, in fact, FIP. To do this, we will need to enforce additional intersections among the sets: given any finite set with the 2-intersection property, we must have that finite set have a nonempty intersection. We will accomplish this by adding a priority system to the construction, allowing us greater control over which B labels occur in which locations.

The new construction is as follows.

We assume that for every i , W_i never enumerates the empty node λ . The set of indices of such W_i is uniformly computable, so this does not change the computability of the construction, and every real trivially meets every W_i that enumerates λ , so meeting or avoiding those that do not is sufficient to be 1-generic.

We use the same A labels as before.

The B labels will now be indexed $B_{i,j,\nu}$, with $i, j \in \omega$, and $\nu \in 2^{<\omega}$. We use these labels in the same way that we used $B_{i,j}$ previously, except that the ν is used to keep track of where higher priority strategies have acted: At stage s , for each $i < s$, in order, if W_i enumerates σ in $\leq s$ steps, then let $\nu \preceq \sigma$ be the longest initial segment of σ such that ν has no extensions τ with a $B_{i'}$ label on it for $i' < i$. If no such ν exists, then do nothing.

(Note, in particular, that this implies that lower priority labels never get placed onto initial segments of nodes that already have higher priority labels, except by the actions described in the next paragraph.) Otherwise, if no initial segments of σ except possibly the empty string λ have a B_i label yet, then for every j such that $j \notin K_s$, we place the label $B_{i,j,\nu}$ onto the node σ .

In addition, however, when we ever place a label B_{i_0,j_0,ν_0} onto a node σ , for every $i < i_0$, for every j , and for every $\nu \preceq \sigma$, we place the label $B_{i,j,\nu}$ on the empty node, λ . (In essence, we discard any label we know we will never use again.)

We build our reals X_i as before, except using this differently labeled tree.

This completes the construction.

We first prove that the modifications introduced achieve the desired result: that at the end of the construction, every 2IP set is FIP.

Lemma 4.1. *Given any finite set, S , of reals from among the reals that we build, those reals have the 2-intersection property if and only if there is a node $\sigma \in 2^{<\omega}$ such that $\tilde{\sigma} \supseteq S$.*

From this, by Lemma 2.1, we may conclude that for this set \mathbb{X} that we build, a set is 2IP if and only if it is FIP.

Proof. Note first that if there is a node $\sigma \in 2^{<\omega}$ such that $\tilde{\sigma} \supseteq S$, then S has the finite intersection property, and so S has the 2-intersection property.

We prove the converse by induction on the size of S , and secondarily by induction on the number of B -labeled sets in S .

If $|S| = 2$, then the lemma holds because the two sets in S intersect if and only if some element was placed into both of them. At the stage when that element was placed into both the sets, it must be because there was some path $\sigma \in 2^{<\omega}$ such that both those sets were in $\tilde{\sigma}$.

If S has no B -labeled set, then the lemma holds because A labels still appear at most once each on $2^{<\omega}$. Thus, either there is some pair of them on incomparable nodes, and so S does not have the 2-intersection

property, or all of their nodes are comparable, and so one of those nodes is an extension of all the other nodes.

If S has one B -labeled set, then the lemma holds because that label must occur on a node that is comparable to the maximal node among those with A labels (otherwise, the S would not have the 2-intersection property). Thus, that maximal node can be extended to (or is already an extension of) a path including a node labeled with that B -label.

Otherwise, let $|S| > 2$, and let B_{i_0, j_0, ν_0} and B_{i_1, j_1, ν_1} be two sets in S . We now consider three cases.

Case 1: $i_0 < i_1$, and at some point, label B_{i_1, j_1, ν_1} is placed on to λ .

Then, by induction, there is a σ such that $\tilde{\sigma} \supseteq (S \setminus \{B_{i_1, j_1, \nu_1}\})$. But then $\tilde{\sigma} \supseteq S$, because $\lambda \preceq \sigma$.

Case 2: $i_0 < i_1$, and B_{i_1, j_1, ν_1} is never placed on to λ .

In this case, first note that because S has 2IP, B_{i_0, j_0, ν_0} and B_{i_1, j_1, ν_1} have nonempty intersection, and so B_{i_1, j_1, ν_1} must at some point be placed on a node that is comparable to a node that already has the label B_{i_0, j_0, ν_0} .

Let σ_0 be a node with the label B_{i_0, j_0, ν_0} that at some point has an extension with the label B_{i_1, j_1, ν_1} . (If B_{i_0, j_0, ν_0} were on an extension of B_{i_1, j_1, ν_1} , then B_{i_1, j_1, ν_1} would be placed on λ .)

Then we have that, $\nu_1 \succeq \sigma_0$ because the ν value of a label is always the by definition of how ν values are chosen when B labels are placed. For the same reason, the label B_{i_1, j_1, ν_1} is only ever placed on extensions of σ_0 . (Instances of B_{i_1} labels that are not on extensions of σ_0 would have a different ν value.) We proceed as in Case 1, but with i_0 and i_1 reversed:

By induction, there is a σ such that $\tilde{\sigma} \supseteq (S \setminus \{B_{i_0, j_0, \nu_0}\})$. But then $\tilde{\sigma} \supseteq S$, because B_{i_1, j_1, ν_1} is only ever placed on extensions of σ_0 , and so $\sigma_0 \preceq \sigma$. By choice of σ_0 , σ_0 has the label B_{i_0, j_0, ν_0} .

Case 3: $i_0 = i_1$.

If either B_{i_0, j_0, ν_0} or B_{i_1, j_1, ν_1} is ever placed onto λ , then we proceed as in Case 1.

If $\nu_0 = \nu_1$, then by symmetry, assume either that j_0 enters K before j_1 or that neither j_0 nor j_1 is in K . Then every node with the label B_{i_0, j_0, ν_0} also has the label B_{i_1, j_1, ν_1} . By induction, we may let σ be such that $\tilde{\sigma} \supseteq (S \setminus \{B_{i_1, j_1, \nu_1}\})$. B_{i_0, j_0, ν_0} is in $\tilde{\sigma}$, and so B_{i_1, j_1, ν_1} is also in $\tilde{\sigma}$.

We now claim that it is not possible that $\nu_0 \neq \nu_1$, and neither B_{i_0, j_0, ν_0} nor B_{i_1, j_1, ν_1} is ever placed onto λ .

Let σ_0 be any path whose labels include B_{i_0, j_0, ν_0} and B_{i_1, j_1, ν_1} . Then ν_0 and ν_1 are each the respective shortest initial segments of σ_0 that had no higher priority extensions when the respective labels were placed. Thus in between when the two labels were placed, one of them, by symmetry ν_0 must have had a higher priority B label placed on an extension of it. But then B_{i_0, j_0, ν_0} is placed on λ , providing a contradiction. \square

With this lemma, we have that for this new construction, the FIP sets are exactly the 2IP sets, because the finite intersection property and the 2-intersection property agree on finite sets.

The construction is defined as before, and we define Φ as before, so every lemma from the previous two sections holds except possibly Lemma 3.1, which is the only Lemma that references the specifics of how the B sets are placed, and also the only lemma that references the specifics of how the Ψ_i were defined. We now define our analogues of the Ψ_i that we will need for this construction.

Let $\Psi_{i, \nu}$ be the functional such that $\Psi_{i, \nu}^S(j) = 1$ if $j \in K$, and $\Psi_{i, \nu}^S(j) = 0$ if $B_{i, j, \nu} \in S$. Formalize this in the same way as for the Ψ_i of the previous construction.

Lemma 4.2. *Let f be FIP (or equivalently 2IP) for \mathbb{X} . Let $Y = \Phi^f$ and assume Y is not 1-generic. Then there exist some i, ν such that $\Psi_{i, \nu}^f$ is a computation K .*

As a reminder, in this construction, we assume that for every i , W_i never enumerates λ .

Proof. Let f be FIP for \mathbb{X} . Let $Y = \Phi^f$, the unique path through $2^{<\omega}$ such that $\tilde{Y} \subseteq \tilde{f}$.

If Y is not 1-generic, then let i_1 be minimal such that Y neither meets nor avoids W_{i_1} . Let i_0 be minimal such that there is no B_{i_0} label on an initial segment of Y besides λ , and such that every initial segment of Y has an extension with a B_{i_0} label. We claim that i_0 exists. To prove this, we show that if every $i < i_1$ does not satisfy the property that i_0 must satisfy, then i_1 satisfies that property:

Assume that every $i < i_1$ does not satisfy this property, and let σ_1 be the shortest initial segment of Y such that for every $i < i_1$ there is no extension of σ_1 with a B_i label. Then, because Y neither meets nor avoids W_{i_1} , there are cofinally many locations along Y where a B_{i_1} label will want to be placed. If those locations are extensions of σ_1 , then that label will actually be placed, because a ν value will be found, and so every initial segment of Y has an extension with a B_{i_1} label. Furthermore Y does not meet W_{i_1} , and so no B_{i_1} label will be placed on an initial segment of Y besides λ .

Therefore, i_0 exists. Let ν_0 be the shortest initial segment of Y such that for every $i < i_1$ there is no extension of ν_0 with a B_i label. We now claim that Ψ_{i_0, ν_0}^f is a computation of K .

To prove this, we first show that for every j , $B_{i_0, j, \nu_0} \notin \tilde{Y}$. By choice of i_0 , we know that B_{i_0, j, ν_0} does not appear on an initial segment of Y besides λ , and by choice of ν_0 , B_{i_0, j, ν_0} is never put on λ .

Next we show that if $B_{i, j, \nu} \notin \tilde{Y}$, then $B_{i, j, \nu}$ occurs on extensions of all initial segments of Y if and only if $i = i_0$, $\nu = \nu_0$, and $j \notin K$.

By assumption, every initial segment of Y has an extension with a B_{i_0} label. If that label was placed after the stage at which every $\sigma \prec \nu_0$ is seen to have an extension with a B_i label for some $i < i_0$, then that B_{i_0} label has ν value ν_0 . If $j \notin K$, then the label B_{i_0, j, ν_0} is placed at that location together with the aforementioned label. Thus, if $i = i_0$, $\nu = \nu_0$, and $j \notin K$, then $B_{i, j, \nu}$ occurs on extensions of all initial segments of Y .

We prove the converse in cases. Assume $B_{i, j, \nu} \notin \tilde{Y}$.

If $i < i_0$, then by choice of i_0 , $B_{i, j, \nu}$ does not occur on extensions of all initial segments of Y .

If $j \in K$, then $B_{i, j, \nu}$ appears at only finitely many locations, and so $B_{i, j, \nu}$ does not occur on extensions of all initial segments of Y .

If $i = i_0$, and $j \notin K$, and $\nu \neq \nu_0$, then after the stage at which every $\sigma \prec \nu_0$ is seen to have an extension with a B_i label for some $i < i_0$, the only ν value used for B_i labels is ν_0 , so $B_{i, j, \nu}$ does not occur on extensions of all initial segments of Y .

Finally, if $i > i_0$, then first note that if $B_{i, j, \nu}$ does not occur on λ , then $B_{i, j, \nu}$ only occurs on extensions of ν , and so if $B_{i, j, \nu}$ occurs on extensions of all initial segments of Y , then $\nu \prec Y$. So then, choose $\sigma \prec Y$ such that $\sigma \succ \nu$. By assumption on i_0 , at some stage, a B_{i_0} label is placed on an extension of σ . At that stage, $B_{i, j, \nu}$ is placed on λ , and so $B_{i, j, \nu} \in \tilde{Y}$, providing a contradiction.

Therefore, we may conclude that $\tilde{f} = \tilde{Y} \cup \{B_{i_0, j, \nu_0} : j \notin K\}$ has the finite intersection property, and no other reals may be added to \tilde{Y} while preserving the finite intersection property.

Thus Ψ_{i_0, ν_0}^f is a computation of K . □

As before, we may now conclude our desired result.

Theorem 4.3. *Let X be a real that can compute an 2IP function for any uniformly computable sequence of reals. Then X can compute a Cohen 1-generic.*

5. REVERSE MATHEMATICS

In this section, we discuss the reverse mathematical consequences of our work from the previous sections. We discuss the reverse mathematical principles FIP and 2IP, defined by Dzhafarov and Mummert in [4], we define the principle “1-GEN,” and we use work from this paper as well as from [2] to prove that FIP and 1-GEN are equivalent over RCA_0 .

In [4], it was proved that FIP implies 2IP over RCA_0 , and our work from Section 4 shows that 2IP also implies 1-GEN, and therefore FIP. However, this implication appears to require $\text{I}\Sigma_2$ for the proof. Our proof utilizes $\text{I}\Sigma_2$ in one location, and $\text{B}\Sigma_2$ in two others.

Definition 5.1 (Dzhafarov, Mummert [4]). *FIP is the principle of 2nd order arithmetic that says:*

“Let \mathbb{X} be a real, thought of as coding the sequence $\langle X_i \rangle$ of reals that are its columns. Assume at least one X_i is nonempty. Then there exists an f such that $\{X_{f(i)}\}$ is maximal with respect to the finite intersection property.”

(Our original definition of an FIP real was over computable values of \mathbb{X} . This statement can thus be thought of as the statement “every \mathbb{X} , computable or not, has an f that is FIP with respect to \mathbb{X} .”)

Definition 5.2 (Dzhafarov, Mummert [4]). *nIP is the same principle of 2nd order arithmetic, except with “finite intersection property” replaced with “n-intersection property.”*

Definition 5.3. *1-GEN is the principle of 2nd order arithmetic that says:*

“For every uniformly Σ_1 collection of sets, there is an X that meets or avoids every set in the collection.”

Note that in practice, this says that for every Y , there exists a X such that X is Cohen 1-generic over Y , because the Σ_1 collection can be a universal Σ_1^Y set, and because, given any uniformly Σ_1 collection of sets, any X that is Cohen 1-generic over the parameters in the definition would necessarily meet or avoid every set in the collection.

We begin by demonstrating the equivalence between FIP and 1-GEN.

In [2], it was proven that any Turing degree that can compute a 1-generic is also an FIP degree. We summarize this proof, observe that the proof relativizes and verify that the details can be carried out in RCA_0 to prove that 1-GEN implies FIP over RCA_0 .

Proposition 5.4. $\text{RCA}_0 \vdash 1\text{-GEN} \rightarrow \text{FIP}$.

Proof (sketch). Given a sequence of sets $\mathbb{X} = \langle X_i \rangle$, let f be a generic for the forcing in which conditions are partial functions g , with finite domain, such that $\{X_{g(i)}\}$ has nonempty intersection.

If f is generic for this forcing, then $\{X_{f(i)}\}$ clearly has the finite intersection property. Furthermore, it is maximal by the following argument.

If, for some n , $\{X_{f(i)}\} \cup \{X_n\}$ has the finite intersection property, then at every stage during the forcing, it would have been possible to extend g to have n in the range of g . In this case, by genericity of f , n must be in the range of f .

This argument is readily formalized in RCA_0 . \square

The converse to Proposition 5.4 follows from the work in this paper.

Proposition 5.5. $\text{RCA}_0 \vdash \text{FIP} \rightarrow 1\text{-GEN}$.

Proof (sketch). The proofs in Sections 2 and 3 are mostly finitary in nature, and so readily formalized in RCA_0 . The infinitary arguments mostly follow from the definitions, and so require no induction. (For instance, a label appears on cofinally many extensions of the path precisely if a given c.e. set of nodes is dense along that path.)

The only use of induction is in Lemma 2.3, which states that an FIP sequence must contain $\{A_\sigma\}$ for σ in a unique infinite path through $2^{<\omega}$. This conclusion requires Σ_0 induction: an infinite path is a collection of nodes that are all comparable including one node at each level. We show that the nodes are all comparable directly, and we show that there is one node at each level because the nodes are downward closed and there cannot be a last node.

This can all be formalized in RCA_0 . \square

In [4], it was proved that FIP implies 2IP over RCA_0 . We sketch the proof for completeness.

Proposition 5.6. [4]

For every $n > m$, $\text{RCA}_0 \vdash n\text{IP} \rightarrow m\text{IP}$.

For every n , $\text{RCA}_0 \vdash \text{FIP} \rightarrow n\text{IP}$.

In particular, $\text{RCA}_0 \vdash \text{FIP} \rightarrow 2\text{IP}$.

Proof (sketch). We sketch the proof that $\text{RCA}_0 \vdash \text{FIP} \rightarrow 2\text{IP}$.

Given $\mathbb{X} = \langle X_i \rangle$, we create a new sequence $\tilde{\mathbb{X}} = \langle \tilde{X}_i \rangle$ where, whenever we see a finite set of the X_i having the 2-intersection property, we take an element, and put that element into every \tilde{X}_i in the corresponding set of \tilde{X}_i , and nowhere else. This ensures that the finite intersection property for $\tilde{\mathbb{X}}$ is equivalent to the 2-intersection property for \mathbb{X} , and hence that any FIP sequence for $\tilde{\mathbb{X}}$ is a 2IP sequence for \mathbb{X} .

The proofs of the other statements are similar. \square

From our construction, we are able to obtain a partial converse, but our construction appears to require Σ_2 bounding.

Proposition 5.7. $RCA_0 + I\Sigma_2 \vdash 2IP \rightarrow 1-GEN$.

Proof (sketch). Lemma 4.1 is a proof that uses induction, and even after simplification appears to require Δ_2 induction. By a result of Slaman [8], Δ_2 induction is equivalent to Σ_2 bounding. The statement in question is the statement that says that given any finite set of labels used in the second construction, those labels have the 2-intersection property if and only if they appear together on some finite path.

A priori, a label being used in the construction is a Σ_1 event, but we may rephrase the lemma to be quantified over all labels that could potentially be used, and only over sets of size 2 or greater. In this case, the lemma would vacuously be true for labels not used in the construction: they neither appear on paths, nor have the 2-intersection property with anything.

Furthermore, the “if” part of the statement can be proven globally without induction: If the labels appear along a finite path, then they have an element in all of them, and so they have the 2-intersection property.

Thus, induction is only used to prove that if the labels have the 2-intersection property then they appear together along some path. This is still a $\Sigma_1 \rightarrow \Sigma_1$ statement, which is a Δ_2 statement.

Lemma 4.2 also requires induction for two reasons, and this is where we use Σ_2 induction (or, more precisely, $L\Pi_2$, which is equivalent to $I\Sigma_2$. For background on induction principles, see Theorem 5.1 of [6]).

First we need a highest priority i_0 that acts infinitely often along extensions of initial segments of Y . Strategy i acting infinitely often along extensions of initial segments of Y is Π_2 :

$$(\forall \sigma \prec Y)(\exists \tau \succ \sigma)(\text{some } B_i \text{ label appears on } \tau \text{ \& no } B_i \text{ label appears on } \sigma.)$$

Next we need to know that after some initial segment of Y , strategy i_0 can stop changing its ν values. This apparently requires Σ_2 bounding: if each $i < i_0$ acts on extensions of only finitely many initial segments of Y , then we need to know that there is some initial segment of Y after which strategy i_0 will no longer need to change its ν value on extensions of that segment, allowing it to act infinitely often with the same labels and achieving its infinitary requirement. This is an instance of Σ_2 bounding. \square

A note on the proof:

It is tempting to attempt to remove the first instance of Σ_2 induction by letting i_1 be any i such that Y neither meets nor avoids W_i , and then claiming that either some higher priority strategy acts infinitely often or we can win with strategy i_1 . Unfortunately, this does not seem to work, because we need the higher priority strategy to act infinitely often and also not to have any even higher priority strategies act infinitely often.

Without Σ_2 induction, one could potentially imagine a cut below which every strategy acts finitely often, and above which every strategy acts infinitely often, but gets injured infinitely often by higher priority strategies that are also above that cut.

Synthesizing the results of this section, we obtain the following implications.

Theorem 5.8.

$RCA_0 \vdash FIP \leftrightarrow 1-GEN$.

$RCA_0 + I\Sigma_2$ proves the following are equivalent.

- FIP
- $1-GEN$.
- nIP (for any value of n)

In particular, the below statements are all equivalent in ω models of RCA_0 . We do not know whether these equivalences hold over RCA_0 in general.

Question 1. *Does RCA_0 prove $FIP \leftrightarrow 2IP$?*

Indeed, we have an entire tower of open questions, although a positive answer to Question 1 would resolve all of them simultaneously.

Question 2. *Do there exist n, m such that RCA_0 proves $nIP \leftrightarrow mIP$? What about $RCA_0 + B\Sigma_2$?*

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